CHAPTER 1 – PROBABILITY

2.1 – Introduction

- Why is statistics needed for?

  Give support and recommendations to decision making process with uncertainty.(is likely to be, is probable that…)

- What do statistics deal with?

  Collection and analysis of data

  Variability - finding a pattern

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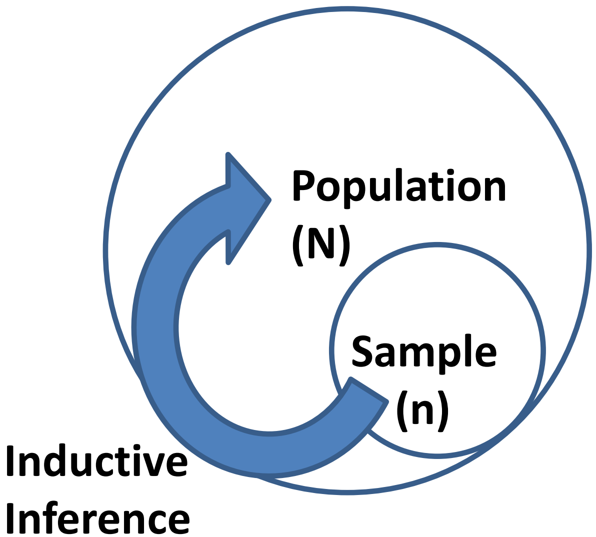
- **Main goal: make statements about the population based in the observation of a sample that have some validity.**



For economic and logistical reasons, to avoid errors of various kinds, only in exceptional cases, the whole population is analysed.

Inductive Inference ⟺ uncertainty

Accuracy ⟺ measure of uncertainty

Population: complete set of items in which the investigator is interested.

Sample: subset of population (randomness)

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- **Probability theory ⟺ uncertainty measurement.**

  How much likely or probable is it?

- **Random experiment:**
  - The outcome cannot be predicted with certainty.
  - The collection of every possible outcome can be described and, in some cases, listed.

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- **Examples of random experiments:**

a) Flip of a coin and observation of head or tails.

b) Toss of a dice and observation of number of spots.

c) A coin is flipped successively at random until the first head is observed and the number of flips counted .

d) Hourly number of phone calls in a call centre.

e) Measure of body fat based on person's weight (kgs) under water.

f) Observation of the daily change (Δ) in an index of stock market prices.

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Definition 2. 1. Sample Space. Event.

Element or sample point (*s*):

Each of the possible outcomes of a random experiment.

Sample Space (S)

Set of all possible outcomes from a random experiment.

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- Sample Spaces can be:

  $a_{1.1}$ **discrete - countable** or **(finite)**

  examples: a) S = {H,T}. $\# S = 2^1$ =2 elements.

  b) S = {1, 2, 3, 4, 5, 6}. $\# S = 6$

  $a_{1.2}$ **discrete - infinite though countable nº of elements**

  examples: c) S=$\{\mathrm{H}, \mathrm{TH}, \mathrm{TTH}, \dots, \mathrm{TT} \dots \mathrm{TH}\}$

  d) S = {0, 1, 2, 3, 4, …}.

  $\boldsymbol{b}$) **continuous** - **uncountable**

  examples: e) S = $\{s: 0 < s < 7\}$.

  f) S = $\{\Delta: 0 < \Delta < 1\}$

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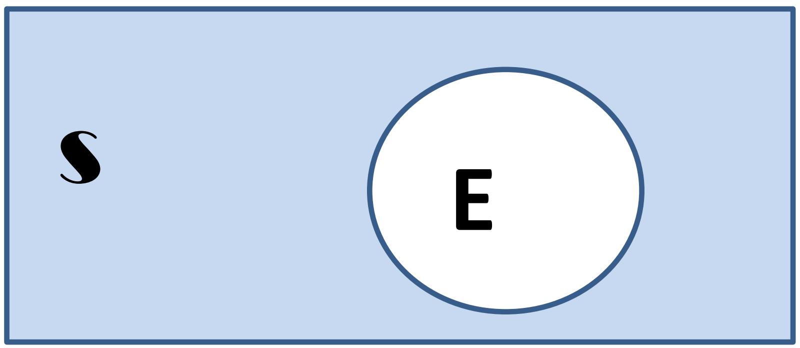
Definition 2. 2. - EVENT.

Any subset of the sample space **S**.



$$\boldsymbol{E} = \{s_1, s_2, \cdots, s_p\} \subset \boldsymbol{S}$$

If E = $\{s\}$ *it is an element.*

$\mathcal{S}$ *is an event too.*

An event $\boldsymbol{E} \subset \mathcal{S}$ occurs if the random experiment results is one of its elements

$$\boldsymbol{s_i \in E \quad i = 1, 2, \cdots, p}$$

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Example: Random Experiment - A coin is flipped twice

Sample space: $\mathbb{S}$ = {HH, HT, TH, TT},

{HH},{HT},{TH},{TT} are elements.

Events (**E** $\subset \mathbb{S}$):

E_1 = {HT,TH} «occurence of exactly one head/tail»;

E_2 = {HH, HT} «occurence of a head in first toss»;

E_3 = {HT,TH,HH} =«occurence of at least one head».

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Example

random experiment: cast of two dices and observation of number of spots in first (i), and second (j) dices.

Sample space: $\mathbb{S}$ = {(i, j): i, j = 1,2,3,4,5,6}.

$\mathbb{S}$ = 6^2 = 36 elements.

Events:

E_1 = {(4,4),(4,5),(5,4),(5,5)}

«occurence of just 4 or 5 spots in both dices»;

E_2 = {(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)}

«total number of spots less than 5».

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Example

random experiment: observation of a light bulb life span (hrs)

Sample space: $\mathbb{S} = \{x : x > 0\} \subset \mathbb{R}$

Events:

$E_1 = \{x : 17000 < x < 43000\}$;
«life span greater than 17000 and lower than 43000 hours »;

$E_2 = \{x : x \leq 50000\}$ «life span less than or equal to 50000 hrs»;

$E_3 = \{x : x > 60000\}$ «life span greater than 60000 hrs».

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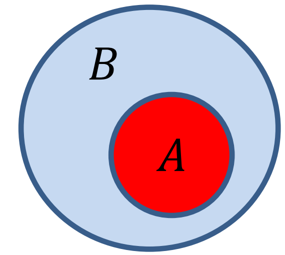
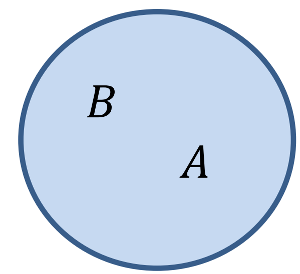
- In the study of probability, sets and events are interchangeable.
- Algebra of events = Algebra of sets



Let A, B be events in sample space $\mathbb{S}$

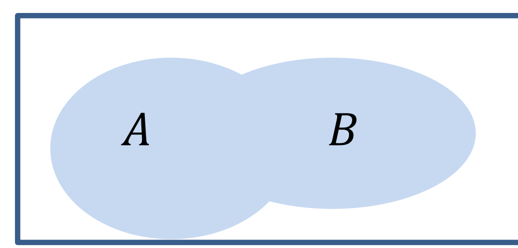
Event A occurs if and only if B occurs. A $\subset$ B.



Event A=Event B if and only if $\forall s \epsilon A \Rightarrow$ s ϵB :

$$A = B \Rightarrow A \subset B \; e \; B \subset A$$

Union of events $\boldsymbol{A} \cup \boldsymbol{B}$
occurs if and only if either A or B or both occur



Null event (∅): event with no elements.

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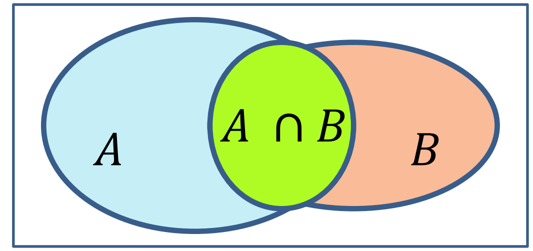
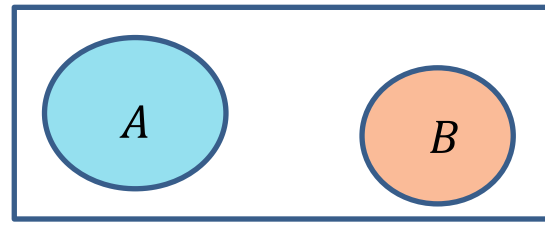
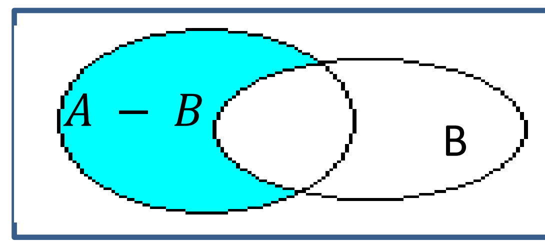
Intersection of events $A \cap B$: is the set of all elements in $\mathbb{S}$ that belong to both A and B



If events A and B have no common elements they are called **Mutually exclusive events.** $A \cap B = \emptyset$

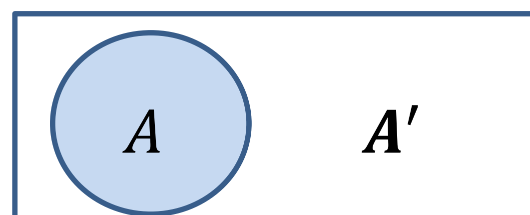


Diference of events $A - B$: is the event that occurs if and only if A occurs but not B.



Complement $\boldsymbol{A'} = \mathbb{S} - A$: the set of elements belonging to $\mathbb{S}$ but not to A. It is obvious that

$$A \cap \boldsymbol{A'} = \emptyset \text{ and } A \cup \boldsymbol{A'} = \mathbb{S}$$



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Some properties of operations over events:

1. Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$

2. Comutativity: $A \cup B = B \cup A$; $A \cap B = B \cap A$

3. Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. De Morgan Laws: $\overline{A \cup B} = \bar{A} \cap \bar{B}$; $\overline{A \cap B} = \bar{A} \cup \bar{B}$

5. *Union of k events* $A_{1,} A_{2,} \cdots, A_k = \cup_{i=1}^{k} A_i$

6. *Intersection of k events* $A_{1,} A_{2,} \cdots, A_k \cdots = \cap_{i=1}^{k} A_i$

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- **Measure of probability. Probability Axioms.**
- **Probable, likely, credible, possible, etc.,**
- **Probability measure the uncertainty degree associated to the occurence of random events .**
- The **measure of the probability** can be formalized through a small number **of postulates.**
- These postulates are due to Bernstein and Kolmogorov decisive contribution. **Kolmogorov Probability postulates .**

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Definition: Measure of probability

Probability of an event is a real valued set function *P that assigns to each event* A *in the sample space* $\mathbb{S}$ *(*$A \subset \mathbb{S}$*) a real number* $P(A)$*, called probability of event* A*, such that the following postulates are satisfied*:

P1 – $P(A) \geq 0.$

P2 – $P(\mathbb{S}) = 1.$

P3 – If $A_1, A_2, \cdots, A_k$ are events satisfying $A_i \cap A_j = \emptyset$, i $\neq j$, then
$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) = P(A_1) + P(A_1) + \cdots + P(A_k)$$
for a finite or an infinite but countable number of events.

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Postulate P3 can be generalized saying that:

- **Theorem 2. 1**– If there is *an infinite but countable number of events*

$$A_{1,} A_{2,} \cdots , A_{n,} \cdots \Rightarrow \cup_{i=1}^{\infty} A_i$$

that are mutually exclusive events,

$$A_i \cap A_j = \emptyset \quad (i \neq j)$$

$$then\ P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

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- **Properties of the measure of probability in a sample space S:**

Theorem 2.3. P($\boldsymbol{A'}$) = 1 − P(A).

Theorem 2.4. $P(\emptyset) = 0\ (*)$.

Theorem 2.5. If $A \subset B \Rightarrow P(A) \leq P(B)$.

Theorem 2.6. $0 \leq P(A) \leq 1$.

Theorem 2.7. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorem 2.8.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

These properties are easily proved.

(*) Null probability of an event does not mean that it can't occur.

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- **Exercise:** A corporation takes delivery of some new machinery that must be installed and checked before it becomes available. The corporation is sure that it will take no more than 7 days for the instalation and check-up to take place.

  Let: $A = \{It\ will\ be\ more\ than\ 4\ days\ before\ the\ machinery\ becomes\ available\}$

  $B = \{It\ will\ be\ less\ than\ 6\ days\ before\ the\ machinery\ becomes\ available\}$

  (a) Describe the event that is the complement of A.

  (b) Are events A and B mutually exclusive?

- **Exercise:** Draw one card from a deck of cards. The sample space is the set of 52 cards.

  Let $A = \{x: x\ is\ a\ jack, queen\ or\ king\}$; $B = \{x: x\ is\ a\ 9\ or\ 10\ and\ red\}$

  $C = \{x: x\ is\ a\ club\}$; $D = \{x: x\ is\ a\ diamond, heart\ or\ spade\}$

  Find: (a) $P(A)$, (b) $P(A \cap B)$, (c) $P(A \cup B)$, (d) $P(C \cup D)$, (e) $P(C \cap D)$.

- **Exercise:** If $P(A) = 0.4$; $P(B) = 0.5$; $P(A \cup B) = 0.7$

  Find $P(A \cap B)$, $P(A - B)$, $P(A')$, $P(A' \cup B')$, $P(A' \cap B')$

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- **Exercise:** Let x equal a number randomly selected from the closed interval $[0, 1]$. Use your intuition to assign values to:

  (a) $P(x\colon 0 \leq x \leq {}^{1}/_{3})$, (b) $P(x\colon {}^{1}/_{3} \leq x \leq 1)$, (c) $P(x\colon x = {}^{1}/_{3})$

  (d) $P(x\colon {}^{1}/_{2} < x < 5)$

- **Exercise**: A point is picked “at random” from a unit square centered in (0,0)

  (a) Define the sample space. $S = \{(x, y)\colon -{}^{1}/_{2} \leq x \leq {}^{1}/_{2}, -{}^{1}/_{2} \leq y \leq {}^{1}/_{2}\}$

  (b) For the probability assignment if $A \in S$, we assign $P(A) = \acute{a}rea_{\,A}$.

  Find $P(A)$:

  $b_1)\ A = \{(x, y)\colon 0 \leq x \leq {}^{1}/_{2}, 0 \leq y \leq {}^{1}/_{2}\}$,

  $b_2)\ A = \{(x, y)\colon (x - {}^{1}/_{2})^2 + (y - {}^{1}/_{2})^2 = 1\}$

  $$b_3)\ A = \begin{Bmatrix} set\ of\ all\ points\ that\ are\ at\ the\ most \\ a\ unit\ distance\ from\ origin \end{Bmatrix}$$

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- Except for the probability of the *null* event (*Ø)* and the *sample space (S)*, probability theory doesn't say how to assign probabilities to events.

- It only says how to find probabilities of certain events from the probabilities of other, usually simpler, events.

- The assignment of probabilities to an event depends on the knowledge about the random experiment and the interpretation that is given to the concept of probability.

- 3 main interpretations of the concept of probability : **classical, relative frequency and subjective.**

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Definitions of the concept of probability .

1. **Classical (De Moivre 1718, Laplace** beggining of XIX century – gambling problems).

   Probability of an even is the proportion of times that the event occurs assuming that all elements in the sample space are equally likelly to occur.

main features:

Sample Space $\mathbb{S}$ is **finite**, $\mathbb{S}$ = $\{s_1 , s_2 , \cdots , s_N\}$

Principle of symmetry

(all outcomes\elements in the sample space **are equally likelly** to occur**).**

$$\boldsymbol{P(s_i) = \frac{1}{N} \quad (i = 1, 2, \cdots , N)}$$

Let event A = $\{s_3 , s_5 , \cdots , s_k\}$ $(k < N)$

$$\boldsymbol{P(A) = \frac{N_A}{N} = \frac{\#A}{\#\mathbb{S}} = \frac{number\ of\ outcomes\ satisfying\ A}{total\ number\ of\ outcomes\ in\ the\ sample\ space}}$$

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- **Example: toss of two "fair" coins, $\mathbb{S}= \{HH, HT, TH, TT\}$.**

  Let, $A = \{occurence\ of\ exactly\ one\ Head\ or\ Tail\} = \{HT, TH\}$;

  $B = \{occurence\ of\ one\ Head\ in\ the\ 1^{st} coin\} = \{HH, HT\}$;

  $C = \{occurence\ of\ at\ least\ one\ tail\} = \{HT, TH, TT\}$;

  Find $P(A)$, $P(B)$, $P(C)$.

- **Example: A manager has a group of eight employees who could be assigned to a project-monitoring task. Three are women and five are men. Two of the men are brothers.**

  Let, $A = \{chosen\ employee\ is\ a\ man\}$;

  $B = \{chosen\ employee\ is\ one\ of\ the\ brothers\}$;

  Find $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$.

- **Consider the course of the Dow-Jones average over two days.**

  Let $\mathbb{S} = \{O_1 , O_2, O_3 . O_4\}$; O_1=avg rises on both days; O_2=avg rises on 1st day but not in 2nd; O_3=avg rises 2nd day but not in 1st; O_4=avg doesn't rise on either day. Let, $A = \{market\ will\ rise\ on\ at\ least\ one\ of\ the\ days\}$.

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- To deal with more complex sample spaces a powerful tool are the Methods of Enumeration : permutations and combinations.

- The classical definition was in rule until the beggining of XX century.

- Two main drawbacks of the classical definition :

  – The principle of symmetry is seldom verified ;
  – Many sample spaces are not finite or countable.

- The **relative frequency definition** tried to overcome the criticisms above. It was adopted since early XX century and is the dominant definition till now.

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Relative frequency definition of probability

- Consider repeating a random experiment $\boldsymbol{n}$ times.

- Count the number of times that event $\boldsymbol{A}$ occured throughout these $\boldsymbol{n}$ repetitions. This number n_A is called the frequency of event $\boldsymbol{A}$.

- The ratio $\frac{n_A}{n}$ is called relative frequency of event $\boldsymbol{A}$.

- The relative frequency is usually very unstable for small values of $\boldsymbol{n}$ but it tends to stabilize for large $\boldsymbol{n}$.

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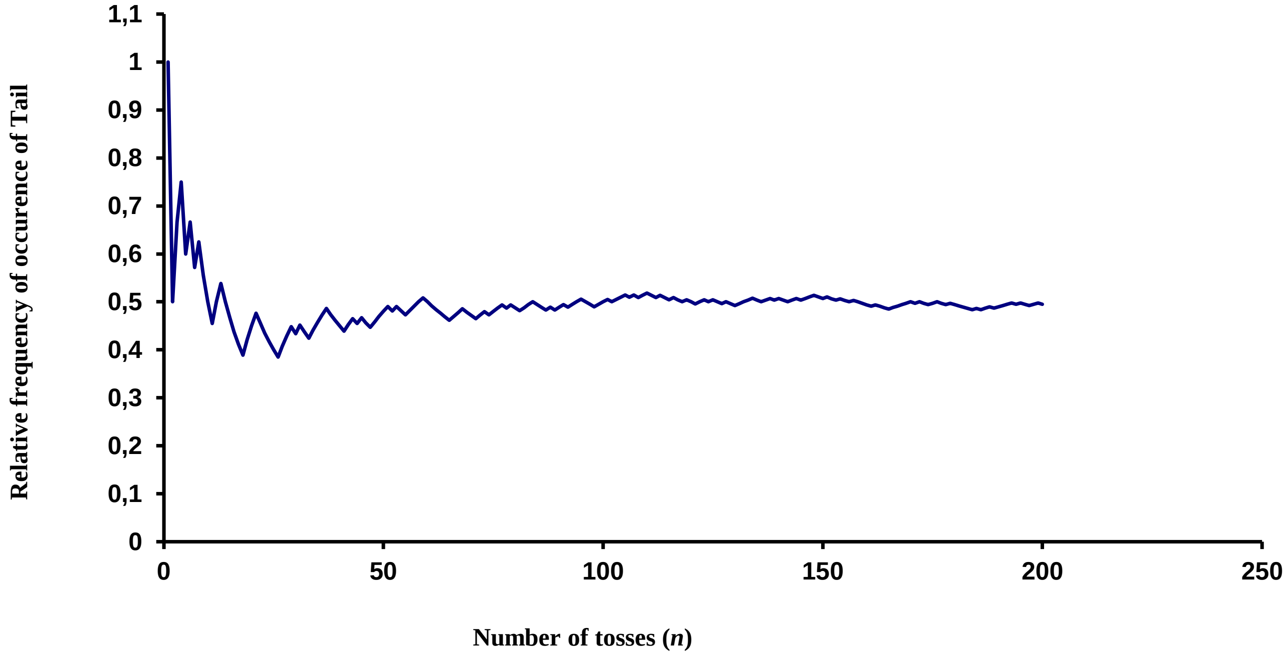
- **Example –** A regular coin is tossed 200 times. The number of times that a Tail occured troughout these $\boldsymbol{n}$ repetitions was counted. The relative frequency of occurence of a Tail calculated for $\boldsymbol{n} = 1, 2, \cdots, n$.



- The probability of event $\boldsymbol{A}$ is the limit of the relative frequency of event $\boldsymbol{A}$ in a large number of trials.

$$P(A) = \lim_{n \to \infty} n_A / n.$$

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- In a more formalized way we can say that:

  For every $\varepsilon > 0$, there is a number n of experiments so that

$$P(A) - \varepsilon < f(A) < P(A) + \varepsilon,$$

is virtually certain

or

$$\lim_{n\to\infty} P[|f(A) - P(A)| < \epsilon] = 1$$

This remarkable result is called the **law of large numbers**.

When ***n*** is set, the relative frequency of an event verify the Kolmogorov postulates.

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- The **subjective definition** of probability deal with the problem of calculating probabilities when the experiment is not symmetric and can't be successively repeated.

- It expresses an individual's degree of belief about the chance that an event will occur

- These subjective probabilities are used in certain management decisions.

  Examples:

  - profitability of a certain project exceed $x\%$.
  - A certain marketing plan will provide in two years a turnover of approximately y thousands euros.
  - The inflation rate next year will be 1%.
  - The exports growth rate in the next quarter will be 2%.

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Subjective probabilities may be determined by finding the odds at which people will consider it fair to bet on the outcome.

Example: A high school principal feels that the odds are 7 to 5 against her getting 100€ raise and 11 to 1 against her getting 200€ raise. Furthermore, she feels that it is an even money bet that she will get one of these raises or the other.

$$P\Big(\underbrace{\text{her getting 100€ raise}}_{A}\Big) = 5/12$$

$$P\Big(\underbrace{\text{her getting 200€ raise}}_{B}\Big) = 1/12$$

$$P\Big(\underbrace{\text{getting one of these raises or the other}}_{A \cup B}\Big) = 6/12 = 5/12 + 1/12$$

So the probabilities are consistent

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- Example: A party official states that the odds are 2 to 1 that he will run for the House of Representatives, and 4 to 1 that he will not run for the Senate. Furthermore, he feels that the odds are 7 to 5 that he will run for one or the other.

$$P\Big(\underbrace{\text{run for the House of Representatives}}_{A}\Big) = 2/3$$

$$P\Big(\underbrace{\text{run for the Senate}}_{B}\Big) = 1/5$$

$$P\Big(\underbrace{\text{run for one or the other}}_{A \cup B}\Big) = 7/12$$

$P(A \cup B) \neq P(A) + P(B)$, so the probabilities are not consistent

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- **Methods of Enumeration**

Theorem 1.1 Multiplication Principle
If an operation consists of two steps, of which the 1st can be done in n_1 ways and for each of these the 2nd one can be done in n_2 ways, then the whole operation can be done in $n_1 \times n_2$ ways.

- Extending to a sequence of more than 2 experiments suppose that a random experiment E_i has $m_i, i = 1,2, \cdots, k$ possible elements.

Theorem 1.2 If an operation consists of k steps of which the 1st can be done in n_1 ways, for each of these the 2nd step can be done in n_2 ways, and so forth, then the whole operation can be done in

$$n_1 \times n_2 \times \cdots \times n_k \text{ elements.}$$

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- **3 important questions** in the choice of the right method of enumeration**:**

1. The order of the selection is noted or not?
2. The sampling is with replacement?
   - replacement occurs when an object is selected and then replaced before the next object is selected.
3. The sampling is without replacement?
   - sampling without replacement occurs when an object is not replaced after being selected.

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Methods of enumeration:

Definition 1.1 Permutations.

A permutation is a distinct arrangement of n different elements of a set

Suppose that $\boldsymbol{n}$ positions are to be filled with $\boldsymbol{n}$ distinct objects.

There are $\boldsymbol{n}$ choices for the 1st position, $\boldsymbol{n}$-1 choices for the 2nd position, ⋯, 1 choice for the last position. So by the multiplication principle, the number of possible arrangements are

$$n(n-1)(n-2)\cdots(2)(1)=n!$$

By definition we take $0!=1$

Theorem 1.3

The number of permutations of n distinct objects is $n!$

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- **Permutation** of n objects (**ordered sampling is without replacement**)

  Example: The number of permutations of the four-letters a,b,c,d $=4! = 24$

- **Permutation** of n objects (**ordered sampling is with replacement**)

  The number of possible arrangements are $\boldsymbol{n^n}$

  Example: The number of possible four-letters code words using a,b,c,d $=4^4 = 256$

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Ordered sampling without replacement

- **Theorem 1.4 Permutation** of $\boldsymbol{n}$ objects taken **r** at a time

If only $\boldsymbol{r}$ positions are to be filled with $\boldsymbol{n}$ distinct objects $\boldsymbol{r} \leq \boldsymbol{n}$, the number of possible **ordered** arrangements is

$$_{n}P_{r} = n(n-1)(n-2)\cdots(n-r+1)! = \frac{n!}{(n-r)!}$$

Example 1: With pieces of cloth of 4 different colours how many distinct three band vertical coloured- flags can one make if the colours can’t be repeated?

$$_{4}P_{3} = \frac{4!}{(4-3)!} = 24$$

Example 2: The number of ways of selecting a president, a vice president, a secretary and a treasurer in a club of 10 persons is

$$_{10}P_{4} = \frac{10!}{(10-4)!} = 5040$$

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Non ordered sampling without replacement

- **Theorem 1.7**

  The number of combinations of $\boldsymbol{n}$ objects taken $\boldsymbol{r}$ at a time is

$$C_k^n = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example 1: In a restaurant there are 10 different menus. How many subsets of 5 different menus can we make from the 10 menus available?

$$C_5^{10} = \frac{10!}{5!(10-5)!} = 36$$

Example 2: The number of possible 5-card hands (in five card poker) drawn from a deck of 52 playing cards is

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} = 2598960$$

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- **Theorem 1.8 Distinguishable Permutation**

  Suppose a set of $\boldsymbol{n}$ objects of $\boldsymbol{r}$ distinguishable types .

  From these $\boldsymbol{n}$ objects, k_1 are similar, k_2 are similar, $\cdots$, k_r are similar, so that $k_1 + k_2 + \cdots + k_r = n$,

  The number of distinguishable permutations of these $\boldsymbol{n}$ objects is:

$$\frac{n!}{k_1! \times k_2! \times \cdots \times k_r!}$$

Multinomial Coeficient

- When r =2 , it is called **Binomial Coeficient** **=**$C_k^n = \frac{n!}{k_1!(n-k_1)!}$.

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- **Distinguishable Permutation**

Example 1: With 9 balls of 3 different colours, 3 black, 4 green and 2 yelow, how many distinguishable groups can one make?

$$\frac{9!}{3! \times 4! \times 2!}$$

Example 2: Suppose that you have a collection of 12 objects, 3 of type A, 4 of type B and 5 of type C. The number of distinguishable permutations of these 12 objects is

$$\frac{12!}{3! \times 4! \times 5!}$$

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- A particularly important application of the methods of enumeration is the resolution of the following problem:
- Consider a population of $\boldsymbol{N}$ elements, from which $\boldsymbol{M}$ has a certain attribute. Suppose that a sample with $\boldsymbol{n}$ elements $(n > 0)$ was selected, what is the probability that there are $\boldsymbol{x}$ $(0 \leq x \leq n)$ elements in the sample with this particular attribute?
- The answer depends on the type of sampling :

Without replacement $\rightarrow$ **hipergeometric model**

or

With replacement $\rightarrow$ **binomial model**

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- **Sampling without replacement $\rightarrow$ hipergeometric model**

number of possible arrangements: Combination of $\boldsymbol{n}$ objects taken $\boldsymbol{n}$ at a time

the population has $\boldsymbol{M}$ elements with a certain attribute and, $N - M$

without it; the sample has $\boldsymbol{x}$ elements with such attribute and

$\boldsymbol{n} - \boldsymbol{x}$ without it. The number of **favorable arrangements** is,

$$\binom{M}{x}\binom{N-M}{n-x} \quad x \geq max\{0, n-(N-M)\} \text{ e } x \leq min\{n, M\}$$

Then the probability that there are $\boldsymbol{x}$ elements in the sample with this particular attribute is:

$$P(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

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- **Sampling with replacement $\rightarrow$ Binomial model**

  The probability that there are $\boldsymbol{x}$ elements in the sample with this particular attribute is:

$$P_r = \frac{M^x (N-M)^{n-x} \binom{n}{x}}{N^n} = \binom{n}{x} \frac{M^x}{N^x} \frac{(N-M)^{n-x}}{N^{(n-x)}} = \binom{n}{x} p^x q^{(n-x)}$$

  where $p = M \;/\; N$ is the proportion of elements in the population with the attribute and $q = 1 - p$ is the proportion of elements in the population without the attribute.

- In the binomial model only the value of $\boldsymbol{p}$ matters. The values of $\boldsymbol{N}$ and M are not important.

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- **Hipergeometric model**

  example: In a lake there are 100 fishes from which 10 are red. If a fisherman take 5 fishes out of the lake, randomly and **without replacement**, what is the probability that none of them are red?

$$P_{red} = \frac{\binom{10}{0}\binom{100-10}{5-0}}{\binom{100}{5}}$$

- **Binomial**

  example : Consider the same lake. If a fisherman take 5 fishes out of the lake, randomly and **with replacement**, what is the probability that two of them are red?

$$P_{red} = \binom{5}{2} 0.1^2 \times 0.9^3$$

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- **Condicional Probability of event A given the sample space $\mathbb{S}$**
- Two dice are cast. We are interested in the sum of spots of both dice. Let event $\boldsymbol{A}$ = « get a sum of 5 spots»

$$P(A) = P(Sum\ of\ spots = 5) = 4/36 = 1/9$$

Let event $\boldsymbol{B}$ be the number of spots in one of the dice is 4.

Once it occurred, probability of event A should be reassessed because the **sample space narrowed** since it is now composed only by all $(4,*)$ or $(*, 4)$. The occurrence of event $\boldsymbol{A}$ will now happen only if the other die has one spot.

$$P(Sum\ of\ spots = 5|\ one\ of\ the\ dice\ got\ 4) = 2/12 = 1/6$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

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- **Definition 2.4. - Condicional Probability**

Let $\boldsymbol{A}$ and $\boldsymbol{B}$, be any two events in a sample space $\boldsymbol{\mathcal{S}}$, $P(\boldsymbol{A}) > 0$. The **conditional probability** of event $\boldsymbol{B}$ given that event $\boldsymbol{A}$ has occurred is denoted by the simbol $P(A|B)$ and is defined by :

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

The **conditional probability is a measure of probability** and verifies the Kolmogorov postulates.

The **conditional probability** can be seen as a reassessement of the probability of an event when there is information about the occurrence of another event. Once one got the new information that event $\boldsymbol{B}$ has occurred, the sample space is no longer $\boldsymbol{\mathcal{S}}$ but a subset of $\boldsymbol{\mathcal{S}}$, associated to the occurrence of event $\boldsymbol{A}$.

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- Previous example:

$$P(B) = \frac{1}{6} + \frac{1}{6} \ \ casos\ (4,*)\ e\ (*,4)$$

$$P(A \cap B) = 2/36\ casos\ (4,1)\ e\ (1,4)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{1/3} = \frac{1}{6}$$

Example – Totoloto has 49 numbers. Playing with 6 numbers the probability of getting a 1st prize is

$$\frac{1}{\binom{49}{6}} = 6!/(49 \times 48 \times \cdots \times 44) = 7{,}15 * 10^{-8}$$

Hitting the first number, the conditional probability of winning the first prize changes to

$$5!/(48 \times 47 \times \cdots \times 44) = 5{,}84 * 10^{-7}$$

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Example : Suppose that we are given 20 tulip bulbs very similar in appearance and told that 8 will bloom early and 13 will be red in accordance with the following combinations:

	Bloom		**Total**
Colour	Early	Late	
Red	5	8	13
Yellow	3	4	7
Total	8	12	20

If one bulb is selected at random, what is the probability that it will produce a red tulip? #Sample space($\mathbb{S}$) =20, number of favorable events=13, so $P(Red) = 13/20$.

Suppose that a close examination of the bulb reveals that it will bloom early. After this information is known, the new sample space (E) is the set of all bulbs that bloom early. $\#E = 8$, number of favorable events=5, so $P(Red|Early) = 5/8$.

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- **Theorem 2. 9. - Multiplication rule:**

  The probability that two events, A and B in sample space S, occur simultaneously is

$$P(A \cap B) = P(A|B) \times P(B) = P(B|A) \times P(A)$$

with $P(B) \neq 0$ and\or $P(A) \neq 0$

This rule can easily be generalized to three or more events.

- **Theorem 2. 10.**

  If A, B and C are any three events in a sample space S such that $P(A \cap B) \neq 0$, then

$$\begin{aligned} P(A \cap B \cap C) &= P[(A \cap B) \cap C] \\ &= P(A \cap B) \times P(C|A \cap B) \\ &= P(A) \times P(B|A) \times P(C|A \cap B) \end{aligned}$$

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- Four cards are to be dealt successively at random and without replacement from a deck of playing cards. The probability of receiving in order a spade (S), a heart (H), a diamond (D) and a club(C) is

$$P(S \cap H \cap D \cap C) =$$
$$= P(S) \times P(H|S) \times P(D|S \cap H) \times P(C|S \cap H \cap D)$$
$$= \frac{13}{52} \times \frac{13}{51} \times \frac{13}{50} \times \frac{13}{49}$$

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- **Definition 2.5. Independence**

Let A and B be any two events. These events are said to be **statistically independent** if and only if

$$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$$

From the multiplication rule it follows that

$$P(A|B) = P(A) \quad if\ P(B) > 0$$

$$P(B|A) = P(B) \quad if\ P(A) > 0$$

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- Toss a coin twice and observe the sequence of head and tails.

$$\mathbb{S} = \{HH, HT, TH, TT\}$$

$A = \{heads\ on\ the\ first\ toss\} = \{HH, HT\}$

$B = \{tails\ on\ the\ second\ toss\} = \{HT, TT\}$

$C = \{Tails\ on\ both\ tosses\} = \{TT\} \subset B \Rightarrow P(C) = 1/4$

$$P(A) = P(B) = \frac{1}{2}$$

$P(B \cap C) = \frac{1}{4} \neq P(B) \times P(C) = 1/8$ so the events B and C are not statistically independent.

$P(A \cap C) = P(\emptyset) = 0 \neq P(A) \times P(C) = 1/8$ events A and C are not statistically independent.

$P(A \cap B) = \frac{1}{4} = P(A) \times P(B)$, so events A and B are statistically independent.

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- **Statistically independent events and mutually exclusive events mean different relations between events:**

- If $P(A) = 0$, then A is an event independent from any other event. Independent events and mutually exclusive events mean the same only and only if the above situation occurs.

- If $P(A) > 0$ e $P(B) > 0$, and events A e B, are mutually exclusive events then A e B are statistically dependent events, since
$$P(A \cap B) = 0 \neq P(A) \times P(B).$$
- **Independence is a property of the probability measure. It is not a property of events.**
- It is not possible to define independence based on events only.

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- **Theorem $\mathbf{2.11}$.** If A and B are statistically independent events, A and B', A' and B, A' and B' are statistically independent events too.

example:

$$P(A \cap B') = P(A - B) = P(A) - P(A \cap B)$$
$$= P(A) - P(A) \times P(B) = P(A)\big(1 - P(B)\big) = P(A) \times P(B')$$

When three events, A, B e C, are considered the following situations may occur :

1 - Events are two by two independent but

$$P(A \cap B \cap C) \neq P(A) \times P(B) \times P(C).$$

2 - $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$ but A and B or A and C or B and C are statistically dependent events .

3 - Events are two by two independent and

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C).$$

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- Example 1: A box has four balls (1, 2, 3,4). One ball is taken out and the number observed. $\mathbb{S} = \{1, 2, 3, 4\} \Rightarrow \#\mathbb{S} = 4$

Let $A = \{1,2\}, B = \{1,3\}, C = \{1,4\}$ be events from Ω

$P(A) = P(B) = P(C) = 1/2;$

$P(A \cap B) = 1/4 = P(A) \times P(B), P(A \cap C) = 1/4 = P(A) \times P(C),$
$P(B \cap C) = 1/4 = P(B) \times P(C)$ but

$P(A \cap B \cap C) = 1/4 \neq P(A) \times P(B) \times P(C) = 1/8$

Example 2: A regular coin is tossed 3 times. Let A={CCC,FCC,CFF,**FFF**}, B={CCF,CFC, FFC, **FFF**} and C={CCF, CFC, FCF, **FFF**} be events from

$\mathbb{S} = \{CCC, CCF, CFC, FCC, CFF, FCF, FFC, FFF\} \Rightarrow \#\mathbb{S} = 8$

$P(A) = P(B) = P(C) = 1/2\,;$
$P(A \cap B \cap C) = 1/8 = P(A) \times P(B) \times P(C) = 1/8$ but
$P(B \cap C) = 3/8 \neq P(B) \times P(C) = 1/4$

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- Example: A regular coin is tossed 3 times.

$$\mathbb{S} = \{CCC, FCC, FCF, FFC, CFC, CCF, FCC, FFF\};\ \#\mathbb{S} = 8$$

Let $A = \{CCC, FCC, CFF, FFF\}$, $B = \{CCF, CFC, FCC, FFF\}$ and $C = \{CCC, FCC, CCF, FFC\}$ be events of the sample space.

$A \cap B = \{FCC, FFF\}$; $A \cap C = \{CCC, FCC\}$; $B \cap C = \{CCF, FCC\}$;

$A \cap B \cap C = \{FCC\}$

Then: $P(A) = P(B) = P(C) = 1/2$

$P(A \cap B) = \frac{1}{4} = P(A) \times P(B)$; $P(A \cap C) = \frac{1}{4} = P(A) \times P(C)$

$$P(B \cap C) = \frac{1}{4} = P(B) \times P(C)$$

$$P(A \cap B \cap C) = \frac{1}{8} = P(A) \times P(B) \times P(C)$$

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Definition 2.6. Independence of more than 2 events

Completelly independent or Mutually independent events

Events A, B e C belonging to the same sample space are **Completelly independent events** if and only if:

$$P(A \cap B) = P(A).P(B), P(A \cap C) = P(A).P(C), P(B \cap C) = P(B).P(C)$$

and

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$$

More generally, $E_1, E_2, \cdots, E_k$ are **mutually statistically independent** if and only if

$$P(E_1, E_2, \cdots, E_k) = P(E_1) \times P(E_2) \times \cdots \times P(E_k) \quad \text{and}$$

$$P(E_i) \times P\left(E_j\right) = 0 \quad \mathrm{i} \neq j \quad (i, j = 1, 2, \cdots, k)$$

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Sample Space Partition

The class of events $\{B_1, B_2, \cdots, B_k, \cdots\}$ is said to be a sample space partition if and only if

$$\cup_j B_j = \mathbb{S} \text{ and } B_i \cap B_j = \emptyset \ (i \neq j), i, j = 1,2, \cdots, k, \cdots$$

then $P(\cup_j B_j) = \sum_j P(B_j) = 1, j = 1,2, \cdots, k, \cdots$

Theorem 2.12 Total probability theorem

If $\{B_1, B_2, \cdots, B_k \cdots\}$ is a partition of sample space $\mathbb{S}$ and

$P(B_j) > 0 \ (j = 1, 2, \cdots, k, \cdots)$, for any event A,

$$P(A) = \sum_j P(A \cap B_j) = \sum_j P(B_j) \times P(A|B_j)$$

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- Bowl 1 contains 2 red and 4 white chips, Bowl 2 contains 1 red and 2 white chips, Bowl 3 contains 5 red and 4 white chips. The probabilities for selecting the bowls are given by $P(B_1) = \frac{1}{3}$, $P(B_2) = \frac{1}{6}$ and $P(B_3) = \frac{1}{2}$.

The class of events $\{B_1, B_2, B_3\}$ is a partition of the sample space if and only if

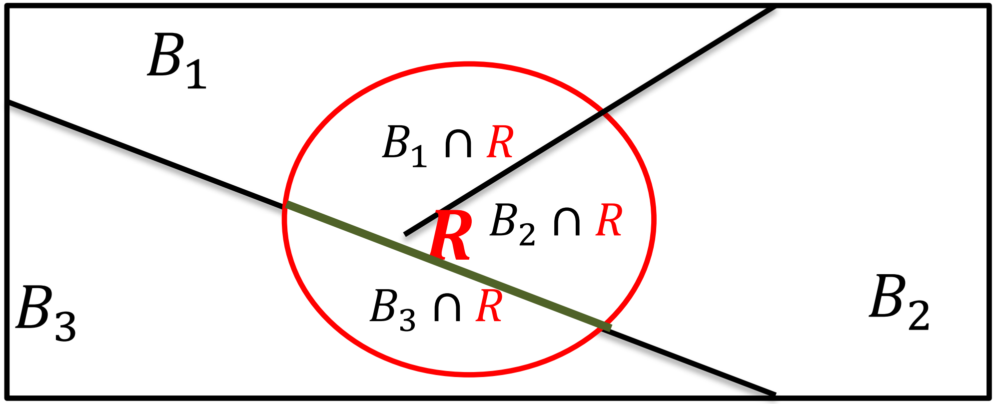
$$B_i \cap B_j = \phi \; i, j = 1, 2, 3$$

and

$$\cup_{j=1}^{3} B_j = \mathcal{S} \Leftrightarrow P(B_1 \cup B_2 \cup B_3) = 1$$

- Consider the experiment selecting a bowl and then drawing a chip at random from the bowl. The probability of drawing a red chip -$P(R)$ is

$$P(R) = P(B_1 \cap R) + P(B_2 \cap R) + P(B_3 \cap R)$$



$$P(R) = P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_3)P(R|B_3)$$

$$P(R) = \frac{1}{3} \times \frac{2}{6} + \frac{1}{6} \times \frac{1}{3} + \frac{1}{2} \times \frac{5}{9} = \frac{4}{9}$$

Is the **total probability of event R**.

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- Now suppose that the outcome of the experiment is a red chip but we don't know from which bowl it was drawn. The conditional probability that the chip was drawn from bowl 1 is

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)} = \frac{P(B_1)P(R|B_1)}{P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_3)P(R|B_3)}$$

$$P(B_1|R) = \frac{\frac{1}{3} \times \frac{2}{6}}{\frac{1}{3} \times \frac{2}{6} + \frac{1}{6} \times \frac{1}{3} + \frac{1}{2} \times \frac{5}{9}} = \frac{1}{4}$$

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- **Theorem 2.13 Bayes Theorem**

If $\{B_1, B_2, \cdots, B_k, \cdots\}$ is a partition of $\mathbb{S}$ and if

$P(B_j) > 0 \ (j = 1, 2, \cdots, k, \cdots)$, for any event A

such that $P(A) > 0$,

$$P(B_j|A) = \frac{P(B_j) \times P(A|B_j)}{\sum_j P(B_j) \times P(A|B_j)}, (j = 1, 2, \cdots, k, \cdots)$$

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- **Example** – A stock market analyst examined the prospects of the shares of a large number of corporations. It turned out that 25% performed much better than the market average(E_1), 50% about the same (E_2), 25% much worse (E_3) ? Forty percent of the stocks that turned out to do much better than the market average were rated a "good buy" (A), as were 20% of those that did about as well as the market and 10% of those that did much worse. What is the probability that a stock rated a "good buy" performed better than the average.

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$P(E_1) = 0{,}25;\ P(A|E_1) = 0{,}4;\ P(E_2) = 0{,}5; P(A|E_2) = 0{,}2$

$P(E_3) = 0{,}25; P(A|E_3) = 0{,}1$

$P(A) = P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)$

$= 0{,}25 * 0{,}4 + 0{,}5 * 0{,}2 + 0{,}25 * 0{,}1 = 0{,}225$

$$P(E_1|A) = \frac{P(E_1 \cap A)}{P(A)} = \frac{P(E_1)P(A|E_1)}{P(A)} = \frac{0{,}25 * 0{,}4}{0{,}225} = 0{,}44(4)$$

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$P(E_j)$	$P(A\|E_j)$	$P(E_j)P(A\|E_j)$	$P(E_j\|A)$	
0,25	0,4	0,1	0,444(4)	
0,5	0,2	0,1	0,444(4)	
0,25	0,1	0,025	0,111(1)	
1		0,225	1	

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Suppose that you are responsible for detecting the source of error when there is a computer system failure. From past experience you know that there are three sources of error: disk drive, computer memory, operating system.

It is known that 50% of errors are due to the disk drive.

From the pattern of the components performance , it is known that the likelihood that a system failure occurs when there is a disk drive error is 60% and the likelihood that a system failure occurs when there is a memory error is 80%.

It is also known that, a failure having been reported , the likelihood that it was due to a memory error is 40% and that the likelihood of a system failure is 60%.

What is the probability that the system fails when there is an operating system error?

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F – system failure

A_j - source of error

A_j	$P(A_j)$	$P(F\|A_j)$	$P(A_j)P(F\|A_j)$	$P(A_j\|F)$
UD	0,5	0,6		
M		0,8		0,4
SO		?		
	1		0,6	1